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Size effect and asymptotic matching approximations in strain-gradient theories of micro-scale plasticity

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Abstract

To explain the size effect found in the testing of plastic behavior of metals on the micrometer scale, four theories of strain-gradient plasticity, representing generalizations of the deformation theory of plasticity, have been developed since 1993—the pioneering original theory of Fleck and Hutchinson in two subsequent versions, the mechanism-based strain-gradient (MSG) plasticity of Gao and co-workers (the first theory anchored in the concept of geometrically necessary dislocations), and Gao and Huang's recent update of this theory under the name Taylor-based nonlocal theory. Extending a recent study of Bažant in 2000 focused solely on the MSG theory, the present paper establishes the small-size asymptotic scaling laws and load–deflection diagrams of all the four theories. The scaling of the plastic hardening modulus for the theory of Acharya and Bassani, based on the incremental theory of plasticity, is also determined. Certain problematic asymptotic features of the existing theories are pointed out and some remedies proposed. The advantages of asymptotic matching approximations are emphasized and an approximate formula of the asymptotic matching type is proposed. The formula is shown to provide a good description of the experimental and numerical results for the size range of the existing experiments (0.5–100 μm).

Keywords: Asymptotic; Strain; Plasticity; Modulus

1. Introduction

Testing of micro-indentation, micro-torsion and micro-bending of copper and other metals on the micrometer scale (see, e.g., in Fleck et al., 1994; Gao et al., 1999a,b), conducted in the early 1990s, revealed a size effect and a significantly stiffer response than predicted by the classical theory of plasticity calibrated on the macro-scale. A similar stiffening was suggested by experiments demonstrating a great increase of yield strength and plastic hardening in nanocomposites (Lloyd, 1994; Kiser et al., 1996). It became clear that the differences, attributed to the effect of geometrically necessary dislocations (Gao et al., 1999a,b), called for a

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new theory. Development of such a theory was pioneered by Fleck and Hutchinson (1993, 1997). They generalized the deformation theory of plasticity by incorporating into it, in a properly invariant manner, the strain-gradient tensor, which was previously explored in the context of elasticity by Toupin (1962) and Mindlin (1965). Introduction of strain gradients inevitably implies the existence of a material characteristic length (Hutchinson, 1997). This in turn implies the presence of a transitional size effect.

A different theory was recently formulated by Gao et al. (1999a,b) and Huang et al. (2000) and named the mechanism-based strain-gradient (MSG) theory of plasticity. That theory had an exciting novel feature—it was supported, under certain simplifying assumptions, on the theory of dislocations (Hirth and Lothe, 1982; Weertman and Weertman, 1964; Cottrell, 1964) and took into account the characteristic spacing of the geometrically necessary dislocations (as described by Nye (1953)), which gives rise to size effect. However, subsequent numerical simulations of Huang et al. (2000), as well as simultaneous asymptotic scaling analysis of Bažant (2000, 2002), showed that some aspects of the MSG theory were questionable. The size of the so-called 'meso-scale cell' providing the linkage between discrete dislocations and a continuum proved in numerical simulations to be indeterminate (Gao and Huang, 2001), while the small-size asymptotic scaling properties were found to be questionable (Bažant, 2000), impairing the representation of the test data for the smallest sizes (1 μm or less) and making the use of asymptotic matching ineffective.

As a remedy, Bažant (2000, 2002) proposed eliminating the strain-gradient tensor from the differential equations of equilibrium, and Gao and Huang (2001) reached independently the same conclusion upon noting from nonlocal finite element simulations that the meso-scale cell size was best considered to be vanishingly small. Huang et al. (2000) further found an ingenious and numerically friendly representation of the strain gradient through a nonlocal integral, and Gao and Huang (2001) named the updated theory the Taylor-based nonlocal theory (TNT), emphasizing that the theory is anchored in G.I. Taylor's classical work on dislocations (Hirth and Lothe, 1982).

While the aforementioned theories represent generalization of the deformation theory (total strain theory) of plasticity, Acharya and Bassani (2000) developed a gradient generalization of the classical incremental theory of plasticity. In that theory, the gradient effect is explained by lattice incompatibility (Bassani, 2001), which is of course related to the geometrically necessary dislocations.

The purpose of this paper is to extend the previous asymptotic analysis of the MSG theory (Bažant, 2000, 2002) to the updated TNT theory as well as to the, by now classical, theories of Fleck and Hutchinson (1993, 1997). The small-size asymptotic scaling laws and load–deflection diagrams will be determined for these theories and applied for developing simple asymptotic matching formulae for the intermediate range that is of interest for practical applications and is explored in testing. Some special cases of asymptotic scaling of the MSG theory which were not explored in Bažant (2000, 2002) will be also clarified. Mutual comparisons of the existing theories, as well as comparisons with the existing test data on the size effect will be made and documented graphically. The scaling of the plastic hardening modulus in Bassani's theory will be also determined, although a full scaling analysis of that theory is beyond the scope of this paper.

2. Scaling of Fleck and Hutchinson's strain-gradient plasticity

Fleck and Hutchinson (1993) pioneered the development of a phenomenological theory for strain-gradient plasticity (SGP). They called their first theory the couple stress theory (denoted by CS). Later Fleck and Hutchinson (1997) improved their theory, calling it the stretch and rotation gradients theory (denoted by SG).

2.1. Fleck and Hutchinson's formulation

In CS and SG theories, the strain energy density W is assumed to depend on the strain-gradient tensor $\boldsymbol{\eta}$ of components $\eta_{ijk} \equiv u_{k,ij}$ as well as the linearized strain tensor $\boldsymbol{\epsilon}$ of components $\epsilon_{ij} \equiv (1/2)(u_{i,j} + u_{j,i})$

(attention is here restricted to small strains). This assumption comes from the classical work of Toupin (1962) and Mindlin (1965), confined to elastic behavior, and is expressed as

$$W = \tfrac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj} + \mu\epsilon_{ij}\epsilon_{ij} + a_1\eta_{ijj}\eta_{ikk} + a_2\eta_{iik}\eta_{kjj} + a_3\eta_{iik}\eta_{jjk} + a_4\eta_{ijk}\eta_{ijk} + a_5\eta_{ijk}\eta_{kji} \tag{1}$$

where λ and μ are the usual Lamé constants and a_n are additional elastic stiffness constants of the material. Similar to the classical theory, Cauchy stress σ_{ij} is defined as $\partial W/\partial\epsilon_{ij}$, and is work-conjugate to ϵ_{ij}. Furthermore, a higher-order stress tensor $\boldsymbol{\tau}$, work-conjugate to the strain-gradient tensor $\boldsymbol{\eta}$, is defined as $\tau_{ijk} = \partial W/\partial\eta_{ijk}$. If W is defined by (1), the constitutive relation is of course linear. So (1) is suitable only for linear isotropic elastic materials. To extend it to general nonlinear elastic materials, a new variable, an invariant named combined strain quantity, $\mathscr{E}$, is introduced by Fleck and Hutchinson (1997); it is defined as a function of both the strain tensor and the strain-gradient tensor, while the strain energy density W is assumed, for general nonlinear elastic material, to be a nonlinear function of $\mathscr{E}$.

To define $\mathscr{E}$, Fleck and Hutchinson (1997) decompose the strain-gradient tensor $\boldsymbol{\eta}$ into its hydrostatic part $\boldsymbol{\eta}^{\rm H}$ and deviatoric part $\boldsymbol{\eta}'$;

$$\eta^{\rm H}_{ijk} \equiv \tfrac{1}{4}(\delta_{ik}\eta_{jpp} + \delta_{jk}\eta_{ipp}); \quad \boldsymbol{\eta}' = \boldsymbol{\eta} - \boldsymbol{\eta}^{\rm H} \tag{2}$$

To simplify the problem, only incompressible materials are considered in the modeling of metals, in which case $\epsilon'_{ij} = \epsilon_{ij}$ and $\eta^{\rm H}_{ijk} = 0$ (which implies deviatoric strain gradient $\eta'_{ijk} = \eta_{ijk}$). Furthermore, Fleck and Hutchinson (1997) introduce the orthogonal decomposition

$$\boldsymbol{\eta}' = \boldsymbol{\eta}'^{(1)} + \boldsymbol{\eta}'^{(2)} + \boldsymbol{\eta}'^{(3)} \tag{3}$$

in which the three tensors are defined in the component form as

$$\eta'^{(1)}_{ijk} = \eta'^{\rm S}_{ijk} - \tfrac{1}{5}(\delta_{ij}\eta'^{\rm S}_{kpp} + \delta_{jk}\eta'^{\rm S}_{ipp} + \delta_{ki}\eta'^{\rm S}_{jpp}) \tag{4}$$

$$\eta'^{(2)}_{ijk} = \tfrac{1}{6}(e_{ikp}e_{jlm}\eta'_{lpm} + e_{jkp}e_{ilm}\eta'_{lpm} + 2\eta'_{ijk} - \eta'_{jki} - \eta'_{kij}) \tag{5}$$

$$\eta'^{(3)}_{ijk} = \tfrac{1}{6}(-e_{ikp}e_{jlm}\eta'_{lpm} - e_{jkp}e_{ilm}\eta'_{lpm} + 2\eta'_{ijk} - \eta'_{jki} - \eta'_{kij}) + \tfrac{1}{5}(\delta_{ij}\eta'^{\rm S}_{kpp} + \delta_{jk}\eta'^{\rm S}_{ipp} + \delta_{ki}\eta'^{\rm S}_{jpp}) \tag{6}$$

Here $\boldsymbol{\eta}'^{\rm S}$ is a fully symmetric tensor defined as

$$\eta'^{\rm S}_{ijk} = \tfrac{1}{3}(\eta'_{ijk} + \eta'_{jki} + \eta'_{kij}) \tag{7}$$

Using the foregoing three tensors $\boldsymbol{\eta}'^{(i)}$, Fleck and Hutchinson (1997) define the combined strain quantity $\mathscr{E}$ as

$$\mathscr{E}^2 = \tfrac{2}{3}\epsilon'_{ij}\epsilon'_{ij} + \ell_1^2\eta'^{(1)}_{ijk}\eta'^{(1)}_{ijk} + \ell_2^2\eta'^{(2)}_{ijk}\eta'^{(2)}_{ijk} + \ell_3^2\eta'^{(3)}_{ijk}\eta'^{(3)}_{ijk} \tag{8}$$

where ℓ_i are three length constants which are given different values in the CS and SG theories (which is the only major difference between these two theories):

$$\text{For CS:}\ \ \ell_1 = 0, \quad \ell_2 = \tfrac{1}{2}\ell_{\rm CS}, \quad \ell_3 = \sqrt{\frac{5}{24}}\ell_{\rm CS} \tag{9}$$

$$\text{For SG:}\ \ \ell_1 = \ell_{\rm CS}, \quad \ell_2 = \tfrac{1}{2}\ell_{\rm CS}, \quad \ell_3 = \sqrt{\frac{5}{24}}\ell_{\rm CS} \tag{10}$$

Here $\ell_{\rm CS}$ is called the material characteristic length.

Based on the combined strain quantity $\mathscr{E}$ as defined, the strain energy density W can be defined as a function of $\mathscr{E}$ instead of ϵ. Then Cauchy stress tensor $\boldsymbol{\sigma}$ and the higher-order stress tensor $\boldsymbol{\tau}$ (couple stress tensor) can be expressed as:

$$\sigma_{ik} = \frac{\partial W}{\partial \epsilon_{ik}} = \frac{\mathrm{d}W}{\mathrm{d}\mathscr{E}} \frac{\partial \mathscr{E}}{\partial \epsilon_{ik}} \tag{11}$$

$$\tau_{ijk} = \frac{\partial W}{\partial \eta_{ijk}} = \frac{\mathrm{d}W}{\mathrm{d}\mathscr{E}} \frac{\partial \mathscr{E}}{\partial \eta_{ijk}} \tag{12}$$

Using (8) and the condition of incompressibility, one has

$$\frac{\partial \mathscr{E}}{\partial \epsilon_{ik}} = \frac{2\epsilon_{ik}}{3\mathscr{E}} \tag{13}$$

$$\frac{\partial \mathscr{E}}{\partial \eta_{ijk}} = \frac{1}{\mathscr{E}} \left(\ell_1^2 \eta'^{(1)}_{lmn} \frac{\partial \eta'^{(1)}_{lmn}}{\partial \eta_{ijk}} + \ell_2^2 \eta'^{(2)}_{lmn} \frac{\partial \eta'^{(2)}_{lmn}}{\partial \eta_{ijk}} + \ell_3^2 \eta'^{(3)}_{lmn} \frac{\partial \eta'^{(3)}_{lmn}}{\partial \eta_{ijk}} \right) = \frac{\ell_{\mathrm{CS}}^2 C_{ijkmnl} \eta_{mnl}}{\mathscr{E}} \tag{14}$$

where C_{ijkmnl} is a six-dimensional constant dimensionless tensor which could be determined from (4)–(6) and (14). Obviously, the CS and SG theories will be characterized by different tensors C, although, for each of them, tensor C is constant, that is, independent of $\boldsymbol{\epsilon}$, $\boldsymbol{\eta}$ and ℓ_{CS}.

For the sake of simplicity, the following power law is assumed for the strain energy density W (Fleck and Hutchinson, 1997):

$$W = \frac{n}{n+1} \Sigma_0 \mathscr{E}_0 \left(\frac{\mathscr{E}}{\mathscr{E}_0} \right)^{(n+1)/n} \tag{15}$$

where Σ_0, $\mathscr{E}_0$ and the strain hardening exponent n are taken to be material constants (and, for hardening materials, $n \geqslant 1$; typically $n \approx 2$–5). Thus (11) and (12) yield the constitutive relations

$$\sigma_{ik} = \frac{2}{3} \Sigma_0 \left(\frac{1}{\mathscr{E}_0} \right)^{1/n} \mathscr{E}^{(1-n)/n} \epsilon_{ik} \tag{16}$$

$$\tau_{ijk} = \Sigma_0 \left(\frac{1}{\mathscr{E}_0} \right)^{1/n} \ell_{\mathrm{CS}}^2 \mathscr{E}^{(1-n)/n} C_{ijklmn} \eta_{lmn} \tag{17}$$

The principle of virtual work yields the following field equations of equilibrium (Fleck and Hutchinson, 1997):

$$\sigma_{ik,i} - \tau_{ijk,ij} + f_k = 0 \tag{18}$$

2.2. Dimensionless variables

To analyze scaling, conversion to dimensionless variables (labeled by an overbar) is needed. Among many possible sets of such variables, the following will be convenient:

$$\bar{x}_i = x_i/D, \quad \bar{u}_i = u_i/D, \quad \bar{\epsilon}_{ij} = \epsilon_{ij}, \quad \bar{\eta}_{ijk} = \eta_{ijk} D, \quad \bar{f}_k = f_k D/\sigma_{\mathrm{N}} \tag{19}$$

$$\bar{\tau}_{ijk} = \tau_{ijk}/(\Sigma_0 \ell_{\mathrm{CS}}), \quad \bar{\sigma}_{ik} = \sigma_{ik}/\Sigma_0, \quad \bar{\mathscr{E}} = \mathscr{E} \tag{20}$$

$$\bar{\eta}'^{(l)}_{ijk} = \eta'^{(l)}_{ijk} D \quad \text{where } l = 1, 2, 3 \tag{21}$$

Here D is the characteristic length of the structure and σ_{N} is the nominal strength. Using these dimensionless variables, the constitutive law of the SGP theory can be rewritten as:

$$\bar{\sigma}_{ik} = \frac{2}{3}\left(\frac{1}{\mathscr{E}_0}\right)^{1/n} \bar{\mathscr{E}}^{(1-n)/n}\bar{\epsilon}_{ik} \tag{22}$$

$$\bar{\tau}_{ijk} = \left(\frac{1}{\mathscr{E}_0}\right)^{1/n} \frac{\ell_{\mathrm{CS}}}{D} C_{ijklmn}\bar{\mathscr{E}}^{(1-n)/n}\bar{\eta}_{lmn} \tag{23}$$

The field equations of equilibrium transform as

$$\partial_i\bar{\sigma}_{ik} - \frac{\ell_{\mathrm{CS}}}{D}\partial_i\partial_j\bar{\tau}_{ijk} + \frac{\sigma_{\mathrm{N}}}{\Sigma_0}\bar{f}_k = 0 \tag{24}$$

where $\partial_i = \partial/\partial\bar{x}_i$, derivatives with respect to the dimensionless coordinates.

After substituting (22) and (23) into (24), we obtain the differential equations of equilibrium in the form:

$$\frac{2}{3}\left(\frac{1}{\mathscr{E}_0}\right)^{1/n}\partial_i(\bar{\mathscr{E}}^{(1-n)/n}\bar{\epsilon}_{ik}) - \left(\frac{\ell_{\mathrm{CS}}}{D}\right)^2\left(\frac{1}{\mathscr{E}_0}\right)^{1/n}\partial_i\partial_j(C_{ijklmn}\bar{\mathscr{E}}^{(1-n)/n}\bar{\eta}_{lmn}) = -\frac{\sigma_{\mathrm{N}}}{\Sigma_0}\bar{f}_k \tag{25}$$

To avoid struggling with the formulation of the boundary conditions, consider first that they are homogeneous, i.e., the applied surface tractions and applied couple stresses vanish at all parts of the boundary where the displacements are not fixed as 0. All the loading characterized by nominal stress σ_{N} is applied as body forces f_k whose distributions are assumed to be geometrically similar; σ_{N} is considered as the parameter of these forces, all of which vary proportionally to σ_{N}. Then the transformed boundary conditions are also homogeneous. In terms of the dimensionless coordinates, the boundaries of geometrically similar structures of different sizes are identical.

When the structure is not at maximum load but is hardening, one must decide which are the σ_{N} values that are comparable. What is meaningful is to compare structures of different sizes for the same dimensionless displacement field $\bar{u}_k$. Thus, the comparable structures will have the same $\bar{\epsilon}_{ik}$ and $\bar{\eta}_{ijk}$.

2.3. Scaling and size effect

The problem of scaling and size effect can now be fully discussed. The limit $D/\ell_{\mathrm{CS}} \to \infty$ is simple because the dimensionless third-order stresses $\bar{\tau}_{ijk}$ vanish and all the equations reduce to the standard field equations of equilibrium on the macro-scale. The combined strain quantity $\mathscr{E}$ reduces to the classical effective strain, and (15) becomes the usual strain energy density function.

The opposite asymptotic behavior for $D/\ell_{\mathrm{CS}} \to 0$ is a little more complex. From (8) we know that when $D/\ell_{\mathrm{CS}} \to 0$,

$$\bar{\mathscr{E}} = \sqrt{\frac{2}{3}\epsilon'_{ij}\epsilon'_{ij} + \frac{1}{D^2}(\ell_1^2\bar{\eta}'^{(1)}_{ijk}\bar{\eta}'^{(1)}_{ijk} + \ell_2^2\bar{\eta}'^{(2)}_{ijk}\bar{\eta}'^{(2)}_{ijk} + \ell_3^2\bar{\eta}'^{(3)}_{ijk}\bar{\eta}'^{(3)}_{ijk})} \propto D^{-1} \tag{26}$$

It is useful to define another dimensionless variable as follows:

$$\overline{H} = \frac{1}{\ell_{\mathrm{CS}}}\sqrt{\ell_1^2\bar{\eta}'^{(1)}_{ijk}\bar{\eta}'^{(1)}_{ijk} + \ell_2^2\bar{\eta}'^{(2)}_{ijk}\bar{\eta}'^{(2)}_{ijk} + \ell_3^2\bar{\eta}'^{(3)}_{ijk}\bar{\eta}'^{(3)}_{ijk}} \tag{27}$$

Obviously $\overline{H}$ is independent of size D, and we have

$$\bar{\mathscr{E}} \approx \frac{\ell_{\mathrm{CS}}}{D}\overline{H} \quad \text{when } D/\ell_{\mathrm{CS}} \to 0 \tag{28}$$

After substituting (28) into (25), we obtain the differential equations of equilibrium in the form:

$$\frac{2}{3}\left(\frac{\ell_{\mathrm{CS}}}{D}\right)^{(1-n)/n}\left(\frac{1}{\mathscr{E}_0}\right)^{1/n}\partial_i(\overline{H}^{(1-n)/n}\bar{\epsilon}_{ik}) - \left(\frac{\ell_{\mathrm{CS}}}{D}\right)^{(1+n)/n}\left(\frac{1}{\mathscr{E}_0}\right)^{1/n}\partial_i\partial_j(C_{ijklmn}\overline{H}^{(1-n)/n}\bar{\eta}_{lmn}) = -\frac{\sigma_{\mathrm{N}}}{\Sigma_0}\bar{f}_k \tag{29}$$

Now we multiply this equation by $(D/\ell_{\mathrm{CS}})^{(n+1)/n}$ and take the limit of the left-hand side for $D \to 0$. This leads to the following asymptotic form of the field equations:

$$\partial_i \partial_j (C_{ijklmn} \overline{H}^{(1-n)/n} \bar{\eta}_{lmn}) = \chi \bar{f}_k, \quad \text{with } \chi = \bar{\mathscr{E}}_0^{1/n} \frac{\sigma_{\mathrm{N}}}{\Sigma_0} \left(\frac{D}{\ell_{\mathrm{CS}}} \right)^{(n+1)/n} \tag{30}$$

Since D is absent from the foregoing field equation (and from the boundary conditions, too, because they are homogeneous), the dimensionless displacement field as well as the parameter χ must be size independent. Thus we obtain the following small-size asymptotic scaling law for Fleck and Hutchinson's theories of gradient plasticity:

$$\sigma_{\mathrm{N}} = \Sigma_0 \chi \bar{\mathscr{E}}_0^{-1/n} \left(\frac{\ell_{\mathrm{CS}}}{D} \right)^{(n+1)/n} \tag{31}$$

or

$$\sigma_{\mathrm{N}} \propto D^{-(n+1)/n} \tag{32}$$

For hardening materials, we have $1 < (n+1)/n \leqslant 2$.

Since the surface loads may be regarded as the limit case of body forces applied within a very thin surface layer, the same scaling law must also apply when the load is applied at the boundaries.

Although the result (32) applies only to the special case of strain energy density function (15), the same analytical technique can be used for general strain energy functions.

Eq. (32) indicates that the asymptotic behavior on the micro-scale depends on the hardening relation on the macro-scale since the macro-strain-hardening exponent n is involved. Moreover, the asymptotic behavior depends only on that exponent. Generally, the present technique can be used for any strain energy function defined in terms of the strain and strain-gradient tensors, even if no combined strain quantity were defined.

For example, a similar technique can also be used for the strain energy density function (1) defined for linear isotropic elastic material for which the combined strain quantity is not used. The constitutive relation in that case is:

$$\sigma_{ik} = \frac{\partial W}{\partial \epsilon_{ik}} = \lambda \delta_{ik} \epsilon_{ll} + 2\mu \epsilon_{ik} = (2\mu \delta_{il} \delta_{km} + \lambda \delta_{ik} \delta_{lm}) \epsilon_{lm} = D_{iklm} \epsilon_{lm} \tag{33}$$

$$\begin{aligned} \tau_{ijk} &= \frac{\partial W}{\partial \eta_{ijk}} = [2a_1 \delta_{il} \delta_{jk} \delta_{lm} + a_2 (\delta_{in} \delta_{jk} \delta_{lm} + \delta_{ij} \delta_{kl} \delta_{mn}) + 2a_3 \delta_{ij} \delta_{lm} \delta_{kn} + 2a_4 \delta_{il} \delta_{jm} \delta_{kn} + 2a_5 \delta_{in} \delta_{jm} \delta_{kl}] \eta_{lmn} \\ &= C'_{ijklmn} \eta_{lmn} \end{aligned} \tag{34}$$

where the dimension of tensor D_{iklm} is that of a stress, and the dimension of tensor C'_{ijklmn} is that of a force. The dimensionless variables defined in (19) can again be adopted. In terms of the dimensionless variables, the differential equations of equilibrium are:

$$\frac{1}{D} \partial_i (D_{iklm} \bar{\epsilon}_{lm}) - \frac{1}{D^3} \partial_i \partial_j (C'_{ijklmn} \bar{\eta}_{lmn}) + \frac{1}{D} \bar{f}_k \sigma_{\mathrm{N}} = 0 \tag{35}$$

For sufficiently large D, the term with $\bar{\boldsymbol{\eta}}$ will vanish and (35) will become the classical differential equation of equilibrium. For the opposite case, for which D is sufficiently small, the term with $\bar{\boldsymbol{\epsilon}}$ will vanish, and we get the following asymptotic behavior:

$$\sigma_{\mathrm{N}} \propto D^{-2} \tag{36}$$

Of course, (36) is a special case of (32) because, for a linear material, the strain hardening exponent $n = 1$.

2.4. Examples

It is instructive to verify the scaling law in (32) for the basic types of experiments. One important test is that of micro-torsion. It was initially the size effect in this test (Fleck and Hutchinson, 1997) what motivated the development of gradient plasticity.

An effective stress measure Σ may be defined as the work conjugate to $\mathscr{E}$:

$$\Sigma = \frac{\mathrm{d}W(\mathscr{E})}{\mathrm{d}\mathscr{E}} \tag{37}$$

A simple power law relationship between Σ and $\mathscr{E}$ may be adopted;

$$\Sigma = \Sigma_0 \mathscr{E}^N \tag{38}$$

Compared with (15), one sees that $N = 1/n$. For the analysis of size effect, the radius of the wire, D, may be chosen as the characteristic dimension (size). The deformation is characterized by the twist per unit length, κ.

For geometrically similar structures of different sizes, we compare the nominal stresses corresponding to the same dimensionless twist $\bar{\kappa} = \kappa D$. The nominal stress σ_N may be defined as T/D^3, where T is the torque. If the CS theory is used, one has

$$\sigma_N = \frac{T}{D^3} = \frac{6\pi}{N+3} \Sigma_0 \bar{\kappa}^N \left\{ \left[\frac{1}{3} + \left(\frac{\ell_{CS}}{D} \right)^2 \right]^{(N+3)/2} - \left(\frac{\ell_{CS}}{D} \right)^{N+3} \right\} \tag{39}$$

For $\ell_{CS}/D \to \infty$,

$$\left[\frac{1}{3} + \left(\frac{\ell_{CS}}{D} \right)^2 \right]^{(N+3)/2} - \left(\frac{\ell_{CS}}{D} \right)^{N+3} \approx \frac{N+3}{2} \left(\frac{D}{3\ell_{CS}} \right)^2 \left(\frac{\ell_{CS}}{D} \right)^{N+3} \tag{40}$$

from which

$$\sigma_N \propto D^{-N-1} = D^{-(n+1)/n} \tag{41}$$

2.5. Small-size asymptotic load–deflection response

For some special cases such as the pure torsion of a long thin wire or the bending of a slender beam, the displacement distribution can be figured out by the arguments of symmetry and the relative displacement profile remains constant during the loading process. For such problems, the asymptotic load–deflection curve for a very small size D can be determined very easily (Bažant, 2000, 2002).

For such loading, all the dimensionless displacements $\bar{u}_k$ at all the points in a structure of a fixed geometry increase in proportion to one parameter, w, such that $\bar{u}_k = w\hat{u}_k$ where $\hat{u}_k$ is not only independent of D but also invariable during the proportional loading process. Parameter w may be defined as the displacement norm, $w = \|\bar{u}_k\|$.

From (4)–(8), we know that

$$\bar{\mathscr{E}} = w\hat{\mathscr{E}} \tag{42}$$

where $\hat{\mathscr{E}}$ is a function of dimensionless coordinates that does not change during the loading process when size D is very small. Substituting this into the dimensionless constitutive law (22) and (23), we have

$$\bar{\sigma}_{ik} = w^{1/n}\hat{\sigma}_{ik}, \quad \bar{\tau}_{ijk} = w^{1/n}\hat{\tau}_{ijk} \tag{43}$$

where $\hat{\sigma}_{ik}$ and $\hat{\tau}_{ijk}$ are constant during the loading process if the size D is small enough. Thus the load f_k can now be expressed as a function of w as well as the size D. Since the first two terms on the left-hand side of (24) are proportional, respectively, to the functions

$$w^{1/n}D^{(n-1)/n}, \quad w^{1/n}D^{-(n+1)/n} \tag{44}$$

one reaches the conclusion that

$$\bar{f}_k \propto w^{1/n} \tag{45}$$

For hardening materials, we have $1/n \leqslant 1$. So the load deflection curve begins with a vertical tangent. We conclude that the small-size asymptotic load–deflection response is similar to the stress-strain relation for the macro-scale, except that the initial elastic response gets wiped out when $D \to 0$.

3. Scaling of mechanism-based strain-gradient plasticity

The first theory that was based on the consideration of geometrically necessary dislocations was the mechanism-based theory of strain-gradient (MSG) plasticity (Gao et al., 1999a,b; Huang et al., 2000), which is a generalization of the incremental theory of plasticity (Jirásek and Bažant, 2002). The scaling of that theory was analyzed in detail in Bažant (2000, 2002). For the sake of comparison, Bažant's analysis will now be briefly reviewed.

3.1. Formulation of MSG theory

In the MSG theory, the strain-gradient tensor $\eta_{ijk} = u_{k,ij}$, as well as its work-conjugate couple stress tensor τ_{ijk}, needs again to be introduced. However, τ_{ijk} is defined in a multiscale framework rather than on the basis of strain energy density (or the potential energy function). The constitutive relation reads:

$$\sigma_{ik} = K\delta_{ik}\epsilon_{nn} + \frac{2\sigma}{3\epsilon}\epsilon'_{ik}, \quad \tau_{ijk} = l_\epsilon^2\left(\frac{K}{6}\eta^{\mathrm{H}}_{ijk} + \sigma\Phi_{ijk} + \frac{\sigma_{\mathrm{Y}}^2}{\sigma}\Psi_{ijk}\right) \tag{46}$$

where

$$\begin{aligned} \Phi_{ijk} &= \frac{1}{\epsilon}(\Lambda_{ijk} - \Pi_{ijk}), \quad \Psi_{ijk} = f(\epsilon)f'(\epsilon)\Pi_{ijk} \\ \epsilon &= \sqrt{\frac{2}{3}\epsilon'_{ij}\epsilon'_{ij}}, \quad \eta = \frac{1}{2}\sqrt{\eta_{ijk}\eta_{ijk}} \end{aligned} \tag{47}$$

and

$$\Lambda_{ijk} = \tfrac{1}{72}[2\eta_{ijk} + \eta_{kji} + \eta_{kij} - \tfrac{1}{4}(\delta_{ik}\eta_{ppj} + \delta_{jk}\eta_{ppi})], \tag{48}$$

$$\Pi_{ijk} = [\epsilon_{ik}\eta_{jmn} + \epsilon_{jk}\eta_{imn} - \tfrac{1}{4}(\delta_{ik}\epsilon_{jp} + \delta_{jk}\epsilon_{ip})\eta_{pmn}]\epsilon_{mn}/54\epsilon^2 \tag{49}$$

$$\eta^{\mathrm{H}}_{ijk} = \tfrac{1}{4}(\delta_{ik}\eta_{jpp} + \delta_{jk}\eta_{ipp}) \tag{50}$$

Here K, elastic bulk modulus; $\epsilon'_{ik} = \epsilon_{ik} - (1/3)\delta_{ik}\epsilon_{nn}$, deviatoric strains; $\epsilon_{ik} = (1/2)(u_{i,k} + u_{k,i})$, strains; $\boldsymbol{\epsilon}$, $\boldsymbol{\eta}$, tensors consisting of components $\epsilon_{ij}, \eta_{ijk}$; η^{H}_{ijk}, volumetric (hydrostatic) part of η_{ijk}. The hardening of the material is defined as $\sigma = \sigma_{\mathrm{Y}}\sqrt{f^2(\epsilon) + l\eta}$, where σ_{Y}, yield stress; σ, ϵ, stress and strain intensities; η, effective strain gradient proportional to the density of geometrically stored dislocations (i.e., to lattice curvature or twist); $f(\epsilon)$, classical plastic hardening function, which reflects the effect of statistically stored dislocations and is an increasing function of a monotonically decreasing slope, $0 < f'(\epsilon) < \infty$. The material intrinsic length l is similar to the parameter $\ell_{\rm CS}$ used in Fleck et al. (1994) and Fleck and Hutchinson (1997) theories, and it is defined by Gao et al. (1999a) as $l = 3\alpha^2(G/\sigma_{\rm Y})^2 b$, where $G =$ shear modulus; α is an empirical constant (usually ranging from 0.2 to 0.5); b is the magnitude of Burgers vector of edge or screw dislocation (e.g., 0.255 nm for copper). Following Bažant (2000, 2002) and similar to Eq. (2.3) of Fleck and Hutchinson (1997), we can consider a more general hardening relation:

$$\sigma = \sigma_{\rm Y}[f^q(\epsilon) + (l\eta)^p]^{1/q} \tag{51}$$

with positive exponents p and q (Gao et al.'s theory corresponds to the case $p = 1, q = 2$); l_ϵ is the size of the so-called "meso-scale cell" (introduced by Gao et al. to set up the higher-order stress tensor τ_{ijk} and the plastic work equality), which is the material length characterizing the transition from standard to gradient plasticity and is interpreted by Gao et al. (1999a) as the minimum volume on which the macroscopic deformation contributions of the geometrically necessary dislocations may be smoothed out by a continuum. l_ϵ is expressed by Gao et al. (1999a) as $l_\epsilon = \beta L_{\rm y} = \beta(G/\sigma_{\rm Y})b$, where $L_{\rm y}$ is the mean spacing between the statistically stored dislocations at yielding, and β is a constant experimental coefficient, suggested by Gao et al. (1999a) to be between 1 and 10. The differential equations of equilibrium (Gao et al., 1999a,b) are the same as (18), although the definitions of the stress, couple stress and load are different.

3.2. Dimensionless variables and scaling analysis

We will again use the dimensionless variables (19) and will further introduce

$$\bar{\epsilon} = \epsilon, \quad \bar{\eta} = \eta D, \quad \bar{\sigma}_{ik} = \sigma_{ik}/\sigma_{\rm Y}, \quad \bar{\tau}_{ijk} = \tau_{ijk}/(\sigma_{\rm Y} l), \quad \bar{\sigma} = \sigma/\sigma_{\rm Y} \tag{52}$$

Since $\eta^{\rm H}_{ijk}$, Λ_{ijk} and Π_{ijk} are defined by Gao et al. (1999a,b) as homogeneous functions of degree 1 of tensors η_{ijk} and ϵ_{ij}, the following dimensionless variables (again labeled by an overbar) may be introduced:

$$\begin{array}{lll} \bar{\eta}^{\rm H}_{ijk} = \eta^{\rm H}_{ijk} D, & \bar{\Lambda}_{ijk} = \Lambda_{ijk} D, & \overline{\Pi}_{ijk} = \Pi_{ijk} D \\ \overline{\Phi}_{ijk} = \Phi_{ijk} D, & & \overline{\Psi}_{ijk} = \Psi_{ijk} D \end{array} \tag{53}$$

The constitutive law may then be rewritten as:

$$\bar{\sigma}_{ik} = \frac{K}{\sigma_{\rm Y}}\delta_{ik}\bar{\epsilon}_{nn} + \frac{2\bar{\sigma}}{3\bar{\epsilon}}\bar{\epsilon}'_{ik}, \quad \bar{\tau}_{ijk} = \frac{l_\epsilon^2}{lD}\left(\frac{K}{6\sigma_{\rm Y}}\bar{\eta}^{\rm H}_{ijk} + \bar{\sigma}\overline{\Phi}_{ijk} + \frac{1}{\bar{\sigma}}\bar{\Psi}_{ijk}\right) \tag{54}$$

and the differential equations of equilibrium become

$$\partial_i\bar{\sigma}_{ik} - \frac{l}{D}\partial_i\partial_j\bar{\tau}_{ijk} + \frac{\sigma_{\rm N}}{\sigma_{\rm Y}}\bar{f}_k = 0 \tag{55}$$

Considering, same as before, the loading to consist solely of the body forces f_k, we may again restrict attention to homogeneous boundary conditions. For $D/l \to \infty$, which also implies $D/l_\epsilon \to \infty$, all the equations reduce to the standard formulation of incremental plasticity, with no size effect. The opposite asymptotic behavior for $D/l \to 0$ is more interesting. From (51) we have $\bar{\sigma} \approx (\bar{\eta}l/D)^{p/q}$ when $D/l \to 0$. After substituting (54) into (55), we obtain the differential equations of equilibrium in the form:

$$\partial_i\left[\frac{K}{\sigma_{\rm Y}}\delta_{ik}\bar{\epsilon}_{nn} + \frac{2}{3\bar{\epsilon}}\left(\frac{l\bar{\eta}}{D}\right)^{p/q}\bar{\epsilon}'_{ik}\right] - \left(\frac{l_\epsilon}{D}\right)^2\partial_i\partial_j\left[\frac{K}{6\sigma_{\rm Y}}\bar{\eta}^{\rm H}_{ijk} + \left(\frac{l\bar{\eta}}{D}\right)^{p/q}\overline{\Phi}_{ijk} + \left(\frac{D}{l\bar{\eta}}\right)^{p/q}\bar{\Psi}_{ijk}\right] = -\frac{\sigma_{\rm N}}{\sigma_{\rm Y}}\bar{f}_k \tag{56}$$

There are five terms on the left-hand side of this equation, and for $D \to 0$ they are, in sequence, of the order of

$$\mathrm{O}(1), \quad \mathrm{O}(D^{-p/q}), \quad \mathrm{O}(D^{-2}), \quad \mathrm{O}(D^{-2-p/q}), \quad \mathrm{O}(D^{-2+p/q}) \tag{57}$$

When $D \to 0$, the fourth term will in general be the dominant one and, in consequence, the asymptotic form of the field equation will be:

$$\partial_i \partial_j (\bar{\eta}^{p/q} \overline{\Phi}_{ijk}) = \chi_1 \bar{f}_k, \quad \text{with } \chi_1 = \lambda^{-2} \frac{\sigma_\mathrm{N}}{\sigma_\mathrm{Y}} \left(\frac{D}{l} \right)^{2+p/q} \tag{58}$$

where $\lambda = l_\epsilon / l$, constant. Since D is present neither in the foregoing field equation nor the boundary conditions (as they are homogeneous), the dimensionless displacement field as well as parameter χ_1 must be size independent. This leads to the following small-size asymptotic scaling law (obtained by Bažant, 2000, 2002);

$$\sigma_\mathrm{N} = \sigma_\mathrm{Y} \chi_1 \lambda^2 \left(\frac{l}{D} \right)^{2+p/q}, \quad \text{and for } \frac{p}{q} \to \frac{1}{2}: \ \sigma_\mathrm{N} \propto D^{-5/2} \tag{59}$$

where the last expression corresponds to Gao et al.'s theory. As discussed by Bažant (2000, 2002), the asymptotic size effect given by (59) is curiously strong. It is 5 times stronger than that for similar LEFM cracks on the macro-scale, which is $\sigma_\mathrm{N} \propto D^{-1/2}$, and 25 times stronger than the typical Weibull size effect, which is roughly $D^{-0.1}$.

There exists a special case in which $\overline{\Phi}_{ijk} = 0$ for all i, j, k. This case, which was not analyzed in Bažant (2000, 2002), occurs for micro-bending of a compressible material. In that case, the fourth term on the left-hand side of (56) will vanish and so one cannot use (58) any more. Based on (57), the dominant term for $D \to 0$ will then be the third one. Then the asymptotic form of the field equation will have the form:

$$\partial_i \partial_j (\bar{\eta}^\mathrm{H}_{ijk}) = \chi_2 \bar{f}_k, \quad \text{with } \chi_2 = 6\lambda^{-2} \frac{\sigma_\mathrm{N}}{K} \left(\frac{D}{l} \right)^2 \tag{60}$$

By the same argument as before, parameter χ_2 is size independent. So, the small-size asymptotic scaling law for this special case is:

$$\sigma_\mathrm{N} = \frac{K}{6} \chi_2 \lambda^2 \left(\frac{l}{D} \right)^2 \quad \text{or} \quad \sigma_\mathrm{N} \propto D^{-2} \tag{61}$$

Another special case (not considered by Bažant (2000, 2002)) arises for micro-bending of an incompressible material ($\eta^\mathrm{H}_{ijk} = 0$). Since again $\overline{\Phi}_{ijk} = 0$ for all i, j, k, both the fourth term and the third term in (56) vanish. As seen in (57), the dominant term for $D \to 0$ will now be the fifth. The asymptotic form of the field equation turns to be:

$$\partial_i \partial_j (\bar{\eta}^{-p/q} \bar{\Psi}_{ijk}) = \chi_3 \bar{f}_k, \quad \text{with } \chi_3 = \lambda^{-2} \frac{\sigma_\mathrm{N}}{\sigma_\mathrm{Y}} \left(\frac{D}{l} \right)^{2-p/q} \tag{62}$$

Here the parameter χ_3 is again size independent, and so the small-size asymptotic scaling law for this special case is:

$$\sigma_\mathrm{N} = \sigma_\mathrm{Y} \chi_3 \lambda^2 \left(\frac{l}{D} \right)^{2-p/q} \quad \text{and for } \frac{p}{q} \to \frac{1}{2}: \ \sigma_\mathrm{N} \propto D^{-3/2} \tag{63}$$

3.3. Small-size asymptotic load–deflection response

As before, consider again the special cases in which the displacement distribution (or relative displacement profile) remains constant during the loading, because of symmetry conditions. This for example occurs for torsion and bending. Consider the case of a torsion test of a long circular fiber, in which, by arguments of symmetry, the tangential displacements must vary linearly along every radius. As stated before, all the dimensionless displacements $\bar{u}_k$ at all the points in a structure of arbitrary but fixed geometry increase in proportion to a parameter w such that $\bar{u}_k = w\hat{u}_k$ where $\hat{u}_k$ is not only independent of D but also invariable during the proportional loading process.

Since η and Φ_{ijk} are homogeneous functions of degree 1 of both $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$, we may write $\bar{\eta} = w\hat{\eta}$ and $\overline{\Phi}_{ijk} = w\widehat{\Phi}_{ijk}$, where $\hat{\eta}$ and $\widehat{\Phi}_{ijk}$ are functions of dimensionless coordinates which do not change during the loading process at any small enough size D.

First, let us focus on the case that the size D is sufficiently small (compared to the deflection w). Then the small-size asymptotic forms (59), (61) and (63) still hold and the general case of asymptotic field equation (59) may be rewritten as

$$\partial_i \partial_j \left(\hat{\eta}^{p/q} \widehat{\Phi}_{ijk} \right) = \hat{f}_k, \quad \text{with } \hat{f}_k = \chi_1 \bar{f}_k w^{-1-p/q}, \ \ \chi_1 = \lambda^{-2} \frac{\sigma_N}{\sigma_Y} \left(\frac{D}{l} \right)^{2+p/q} \tag{64}$$

From the field equation, it follows that if the relative displacement distribution (profile) $\hat{u}_k$ is constant during the loading process, as in pure bending of a slender beam or in torsion of a long cylinder, then the body force distribution $\hat{f}_k$ at the small-size limit must be constant as well. Using (64), we see that

$$\bar{f}_k = (\hat{f}_k/\chi_1)\, w^{1+p/q} \quad \text{or} \quad \bar{f}_k \propto w^{1+p/q} \tag{65}$$

because $\hat{f}_k/\chi_1$ is constant during loading. Therefore, for the MSG theory, the asymptotic load–deflection diagram is generally of the type:

$$\bar{f}_k \propto w^{3/2} \tag{66}$$

i.e., the slope is initially zero and then gradually increasing during loading.

However, a special consideration must be given to the case that not only $D \to 0$ but also $w \to 0$, i.e., the beginning of the load–deflection diagram at $D \to 0$. For this limit case, we must start from the original dimensionless field equation (56) rather than its asymptotic form. The five terms that are summed on the left-hand size of (56) are proportional, respectively, to the functions

$$w, \quad (w/D)^{p/q}, \quad w/D^2, \quad w^{1+p/q}/D^{-p/q-2}, \quad w^{1-p/q}D^{p/q-2} \tag{67}$$

For the MSG theory ($p = 1$, $q = 2$), these functions are

$$w, \quad (w/D)^{1/2}, \quad w/D^2, \quad w^{3/2}/D^{-5/2}, \quad w^{1/2}D^{-3/2} \tag{68}$$

and (56) leads to the expression

$$-\frac{\sigma_N}{\sigma_Y}\bar{f}_k = a_1 w + a_2 (w/D)^{1/2} + a_3 w/D^2 + a_4 w^{3/2} D^{-5/2} + a_5 w^{1/2} D^{-3/2} \tag{69}$$

where parameters a_1, a_2, a_3, a_4 and a_5 are independent of D and w, and are constant during loading. When w/D is sufficiently large, the dominant term is $a_4 w^{3/2} D^{-5/2}$, which leads to (66) and the load–deflection curve in Fig. 1(a). But when w/D is sufficiently small, the dominant term is $a_5 w^{1/2} D^{-3/2}$, and so

$$\bar{f}_k \propto w^{1/2} \quad \text{for } w \ll D \tag{70}$$

Hence, the load–deflection curve at $D \to 0$ must begin with a vertical tangent (Fig. 1a and b). The load–deflection curves for $w \ll D$ and $w \gg D$ (Fig. 1a) intersect at point $w = w_0$, which can be solved from $a_4 w_0^{3/2} D^{-5/2} = a_5 w_0^{1/2} D^{-3/2}$, that is

$$w_0 = (a_5/a_4)D \quad \text{or} \quad w_0 \propto D \tag{71}$$

Furthermore, in the special case that the fourth term and the third term in (56) vanish, (56) leads to the expression

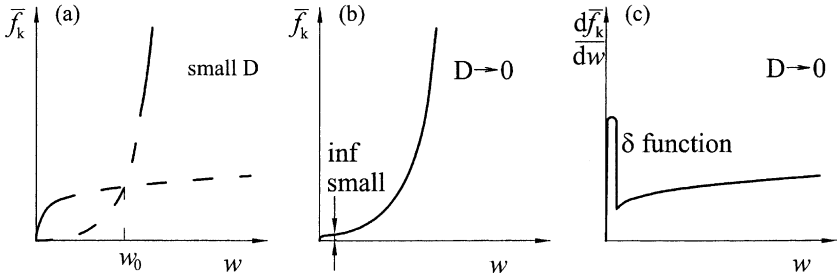

Fig. 1. Load–deflection curves for the general case of the MSG theory: (a) initial and final curves for very small but finite size D, (b) asymptotic load–deflection curve for vanishing size D, and (c) asymptotic curve of load–deflection slope versus deflection, beginning with a Dirac delta function spike.

$$-\frac{\sigma_N}{\sigma_Y}\bar{f}_k = a_1 w + a_2 (w/D)^{1/2} + a_5 w^{1/2} D^{-3/2} \tag{72}$$

Obviously, when w and D are small, the dominant term will in that case be the fifth, regardless the relative ratio of w/D, and so

$$\bar{f}_k \propto w^{1/2} \quad \text{for all } w \tag{73}$$

Note that the load–deflection curves in (70) and (73) begin with a vertical tangent, i.e., the elastic part of response is lost in the asymptotic case.

3.4. Examples

An explicit formula in terms of an integral can be easily obtained for the case of a circular fiber of radius D subjected to torque T; see Eq. (35) in Huang et al. (2000). After transforming that formula to dimensionless coordinates, one has (for $p = 1$ and $q = 2$):

$$\sigma_N = \frac{T}{D^3} = \sigma_Y \frac{2\pi\bar{\kappa}}{3} \int_0^1 \left\{ \frac{\bar{\sigma}}{\bar{\epsilon}} \left(\rho^2 + \frac{l_\epsilon^2}{12D^2} \right) + \frac{l_\epsilon^2 f(\bar{\epsilon}) f'(\bar{\epsilon})}{12D^2 \bar{\sigma}} \right\} \rho \, d\rho \tag{74}$$

where $\bar{\kappa} = \bar{\eta} = \kappa D$, dimensionless specific angle of twist; κ is the actual actual specific angle of twist (rotation angle per unit length of fiber). Taking the limit of $\sigma_N D^{5/2}$ for $D \to 0$, with σ_N given by the foregoing expression, it can be readily checked that the small-size asymptotic form of this formula is

$$\sigma_N = \sigma_Y \left(\lambda^2 l^{5/2} \frac{\pi}{18} \int_0^1 \frac{\rho}{\bar{\epsilon}} d\rho \right) \bar{\kappa}^{3/2} D^{-5/2} \tag{75}$$

This verifies the previous result in (59), as well as (66), with $\bar{\kappa}$ playing the role of $\hat{w}$.

The numerical result of this example can be used to check our asymptotic-matching approximation (109). Following Huang et al.'s (2000) assumption, the hardening relation used is a simple power law relation:

$$f(\epsilon) = \left(\frac{E\epsilon}{\sigma_Y} \right)^N \tag{76}$$

Here E is Young's modulus and N is the plastic work-hardening exponent ($0 \leqslant N < 1$). In numerical computations, $N = 0.2$. The material length l is 4.896 μm. Another length parameter is l_ϵ which depends on the choice of β, $G = 200\sigma_Y$, and in view of definition of l_ϵ (if $\beta = 1$), l_ϵ is 51 nm. According to (75), $s = 5/2$, and from (109) one gets

$$\sigma_0 = \bar{\kappa}^{0.2}, \quad D_0 = 78.74\bar{\kappa}^{0.12}\beta^{0.8} \tag{77}$$

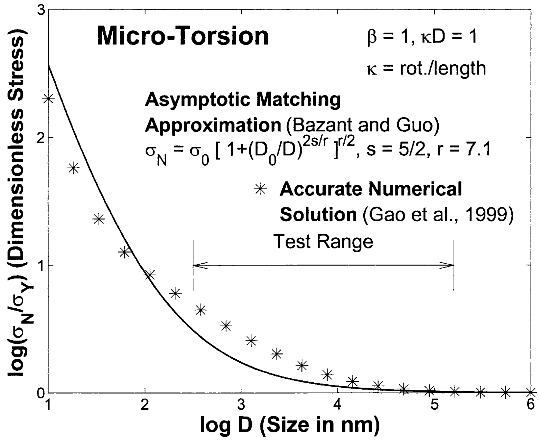

Fig. 2. Gao et al.'s (1999a,b) numerical solution of size effect in micro-torsion based on their MSG theory with $\beta = 1$ (data points), and its optimum fit with the proposed asymptotic matching formula (109) based on the presently determined small-size asymptote of that theory (power law of exponent −5/2).

if $\beta = 1$ and $\kappa = 1$. Parameter r is obtained by optimizing the fit of the data; $r = 7.1$. Fig. 2 shows the optimum fit that is obtained.

From (77), we know that the parameters D_0 and σ_0 are not material properties, they depend on the geometry, deformation, as well as the material constitutive relation.

Another special case for which an explicit formula in terms of an integral can be obtained is the bending of ultra thin nickel beams, which was tested on nickel; see Eq. (29) in Huang et al. (2000). After transformation to dimensionless coordinates, that formula (for $p = 1$ and $q = 2$) reads:

$$\sigma_N = \frac{M}{D^2} = 2\sigma_Y \int_0^{1/2} \left[\frac{2}{\sqrt{3}} \bar{\sigma}\rho + \frac{\bar{\kappa} l_\epsilon^2 f(\bar{\epsilon}) f'(\bar{\epsilon})}{9D^2\bar{\sigma}} \right] \mathrm{d}\rho \tag{78}$$

where D is the beam depth, κ is the bending curvature, and $\bar{\kappa} = \bar{\eta} = \kappa D$. Taking $\lim \sigma_N D^{3/2}$ for $D \to 0$, one finds that the small-size asymptotic form of this formula is

$$\sigma_N = \sigma_Y \left(\lambda^2 l^{3/2} \frac{2}{9} \int_0^{1/2} f(\bar{\epsilon}) f'(\bar{\epsilon})\, \mathrm{d}\rho \right) \bar{\kappa}^{1/2} D^{-3/2} \tag{79}$$

Now we see that this formula is a special case of (63), as well as (73). As pointed out before, (63) applies only to the cases in which the fourth and third terms on the left-hand side vanish. For micro-bending, this condition can be verified analytically. From Eqs. (24) to (27) in Huang et al. (2000), it is easy to find that

$$\Phi_{ijk} = \frac{1}{\epsilon} (\Lambda_{ijk} - \Pi_{ijk}) = 0, \quad \text{for any } i, j, k \tag{80}$$

This means that the fourth term vanishes. Moreover, since the material is assumed to be incompressible, the third term vanishes, too.

Using the same material constitutive relation, as well as the same values of l and l_ϵ, the numerical results for micro-bending are fit as shown in Fig. 3. According to (79), $s = 3/2$. The fit is optimum if r is set to be 10.4, and the parameters σ_0 and D_0 are

$$\sigma_0 = 0.236\bar{\kappa}^{0.2}, \quad D_0 = 5.305\bar{\kappa}^{-2/30}\beta^{4/3} \tag{81}$$

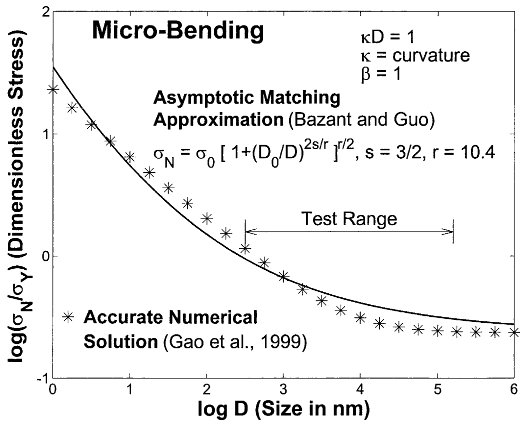

Fig. 3. Gao et al.'s (1999a,b) numerical solution of size effect in micro-bending based on their MSG theory with $\beta = 1$ (data points), and its optimum fit with the proposed asymptotic matching formula (109) based on the presently determined small-size asymptote of that theory (power law of exponent −3/2).

3.5. Discussion

In MSG theory, two extra length parameters are introduced—the material length l and the meso-scale cell size l_ϵ. The material length l is said to be a material property, characterized by the empirical constant α. The presence of l is dictated by dimensional consistency, because of the strain gradient (a similar parameter, ℓ_{CS}, was also introduced in both Fleck and Hutchinson's theories).

The second length parameter l_ϵ defines the size of the meso-scale cell, which is needed to express the couple stress tensor τ_{ijk} and the plastic work equality. According to the definition, l_ϵ is proportional to the constant experimental coefficient β. According to Gao et al. (1999a) and Huang et al.'s (2000) suggestion, parameter β can be between 1 and 10. Then it is interesting to mention that the couple stress τ_{ijk} depends on β (see (46)). Since η_{ijk} is independent of l_ϵ, as well as β, we know that τ_{ijk} should be proportional to β^2. But, at the same time, σ_{ik} is independent of β, as seen in (46).

So the value of β needs to be determined by fitting experimental data. However, as Huang et al. (2000) oberved, "the meso-scale cell size l_ϵ has little effect on the global physical quantities, although it can affect the local deformation field". In the case of micro-torsion, the results for $\beta = 1$ and 10 are almost the same as those for the range of available test data (Fig. 4). This is also true for micro-bending (Fig. 5). So the question arises: Why the couple stress is needed at all when it is found that the couple stress does not affect the global physical quantities?

To answer this question, let us look first where the couple stress comes from. In Gao et al.'s multiscale framework, the stress and strain on the micro-scale (denoted as $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\epsilon}$) obey the Taylor hardening relation:

$$\tilde{\sigma} = \sigma_Y \sqrt{f^2(\tilde{\epsilon}) + l\eta} \tag{82}$$

The micro-scale and meso-scale are linked by the plastic work equality:

$$\int_{V_{\text{cell}}} \tilde{\sigma}_{ik}\tilde{\epsilon}_{ik}\,\mathrm{d}V = (\sigma_{ik}\delta\epsilon_{ik} + \tau_{ijk}\delta\eta_{ijk})V_{\text{cell}} \tag{83}$$

To obtain the constitutive relation, the micro-scale strain in the meso-scale cell is assumed to be linearly distributed:

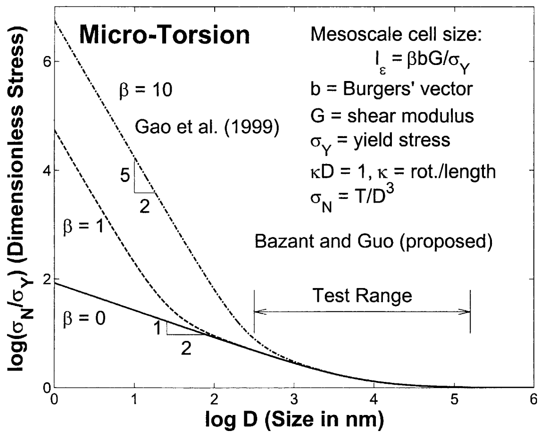

Fig. 4. Size effect plots of the exact solution for micro-torsion based on the Gao et al.'s MSG theory, for different value of parameter β.

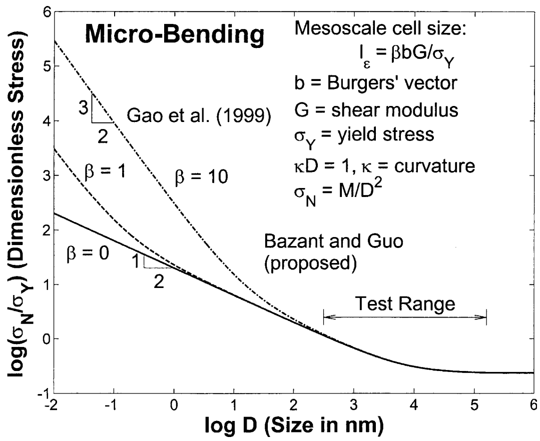

Fig. 5. Size effect plots of the exact solution for micro-bending based on the Gao et al.'s MSG theory, for different value of parameter β.

$$\delta\tilde{\epsilon}_{ij} = \delta\epsilon_{ij} + \frac{1}{2}(\delta\eta_{kij} + \delta\eta_{kji})x_k \tag{84}$$

Then the stress σ_{ik} can be computed as follows:

$$\sigma_{ik} = \frac{1}{V_{\text{cell}}}\int_{V_{\text{cell}}} \tilde{\sigma}_{ik}\,\mathrm{d}V \tag{85}$$

If the entire system is subdivided into many meso-scale cells to allow numerical integration, we find from (84) and (85) that

$$\sum_{\text{All cells}} \int_{V_{\text{cell}}} \tilde{\sigma}_{ik}\tilde{\epsilon}_{ik}\,\mathrm{d}V \approx \sum_{\text{All cells}} \sigma_{ik}\epsilon_{ik}V_{\text{cell}} \tag{86}$$

This formula means that the plastic work can be balanced over the entire system even without being balanced in every cell. Thus there appears to be no practical advantage in introducing the couple stress τ_{ijk} because the plastic work balance in the meso-scale cell need not be considered.

In this regard, it should further be noted that when the couple stress τ_{ijk} is introduced, it is impossible in this theory to define a strain energy density function $W = W(\boldsymbol{\epsilon}, \boldsymbol{\eta})$ such that $\boldsymbol{\sigma} = \partial W/\partial\boldsymbol{\epsilon}$ and $\boldsymbol{\tau} = \partial W/\partial\boldsymbol{\eta}$ (Gao et al., 1999a). The reason is that the constitutive equations of MSG theory do not satisfy the reciprocity relation

$$\frac{\partial\sigma_{ij}}{\partial\eta_{kmn}} \neq \frac{\partial\tau_{kmn}}{\partial\epsilon_{ij}} \tag{87}$$

This implies that the introduction of τ_{ijk} is theoretically disadvantageous.

The effect of β is quite large. If $\beta = 0$, the MSG will change totally. The case $\beta = 0$ means that $l_\epsilon = 0$, and according to the constitutive equation (46), τ_{ijk} will be zero. In other words, there will be no couple stress anymore. Thus the field equation (18) simplifies because the second term on the left-hand side can be omitted, and turns to be the same as that for classical plasticity.

First let us discuss the effect of this simple function on the asymptotic behavior. From (56), we know the last three terms on the left-hand side of (56) will vanish (actually, they stem from the vanished term of the field equation (18)). Thus, for $D \to 0$, the dominant term will be the second one. So the asymptotic form of the field equation reads:

$$\partial_i\left(\bar{\eta}^{p/q}\frac{\bar{\epsilon}'_{ijk}}{\bar{\epsilon}}\right) = \chi_4\bar{f}_k, \quad \text{with } \chi_4 = -\frac{3}{2}\frac{\sigma_{\mathrm{N}}}{\sigma_{\mathrm{Y}}}\left(\frac{D}{l}\right)^{p/q} \tag{88}$$

As before, χ_4 must be size independent. Thus when $\beta = 0$, the small-size asymptotic scaling law for MSG theory turns out to be:

$$\sigma_{\mathrm{N}} = -\sigma_{\mathrm{Y}}\chi_4\left(\frac{l}{D}\right)^{p/q} \tag{89}$$

According to Gao et al.'s theory, $p/q = 1/2$, and so

$$\sigma_{\mathrm{N}} \propto D^{-1/2} \tag{90}$$

We conclude that, upon setting $\beta = 0$, the asymptotic scaling becomes more reasonable.

The change in asymptotic scaling will of course affect the approximate asymptotic matching formula. According to (90), for $\beta = 0$, the parameter s should be 1/2. The numerical results for the case of micro-torsion and micro-bending are now both fit using $\beta = 0$. It is interesting that the parameter r, which would normally be determined by optimum data fitting, is exactly 1 in both examples (the error is within 0.02%, see Figs. 6 and 7). This result is true for other values of $\bar{\kappa}$ and other kinds of examples. So the approximation formula can be written in the following simpler form:

$$\sigma_{\mathrm{N}} = \sigma_0\sqrt{1 + \frac{D_0}{D}} \tag{91}$$

where the parameters σ_0 and D_0 can also be computed according to (110), with $s = 1/2$.

When $\beta = 0$, the small-size asymptotic load–deflection response will also become simpler. Obviously, only the first two terms in (67) will appear because the last remaining three terms will vanish. Then, when D and w are both sufficiently small, the dominant term will be the second one, $(w/D)^{p/q}$, because $p/q \leqslant 1$. Hence,

$$\bar{f}_k \propto w^{p/q} \tag{92}$$

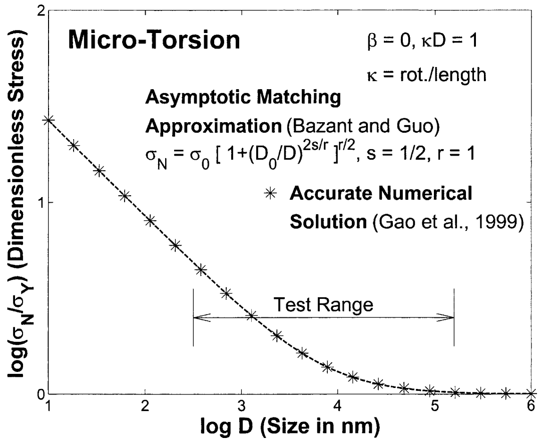

Fig. 6. Gao et al.'s (1999a,b) numerical solution of size effect in micro-torsion based on the special case of MSG theory for $\beta = 0$ (equivalent to TNT theory), and its optimum fit with the proposed asymptotic matching formula (91) based on the small-size asymptotic power law of exponent $-1/2$.

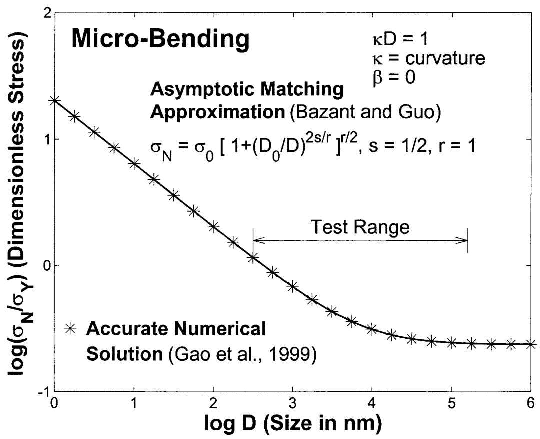

Fig. 7. Gao et al.'s (1999a,b) numerical solution of size effect in micro-bending based on the special case of MSG theory for $\beta = 0$ (equivalent to TNT theory), and its optimum fit with the proposed asymptotic matching formula (91) based on the small-size asymptotic power law of exponent $-1/2$.

Similar to (70) and (73), the elastic part of response asymptotically disappears.

3.6. Tests of micro-hardness

Gao et al. (1999b) showed that the test results for Rockwell micro-hardness tests of copper can be well approximated as

$$\sigma_N = H_0\sqrt{A + \frac{h^*}{D}} \tag{93}$$

where D, depth of indentation; and h^*, A, H_0, empirical constants. This formula, however, is purely empirical. The same data can be also fit very well using another formula, the asymptotic matching

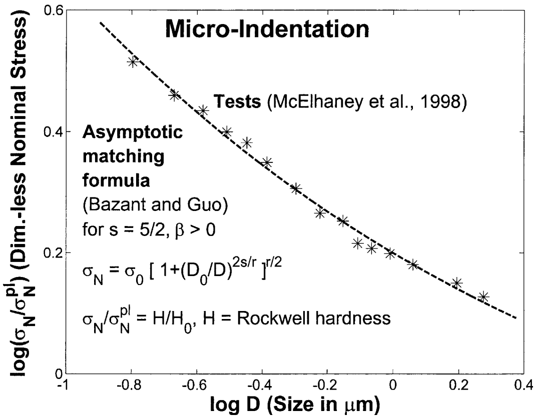

Fig. 8. Data points measured in micro-indentation test of McElhaney et al. (1998), compared to optimum fit with the proposed asymptotic matching formula (109) based on the presently determined small-size asymptote of MSG theory, representing a power law of exponent −5/2.

formula (109) given later, where s is set as 2.5 to fit Gao et al.'s MSG theory (Fig. 8). For $s = 1/2$, formula (109) becomes identical to the last equation (which, by coincidence, happens to have the same form as the formula for the deterministic size effect due to a boundary layer of distributed cracking in flexure of unreinforced concrete beams; Bažant and Planas, 1998).

4. Scaling of Taylor-based nonlocal theory of plasticity

Similar to observation in Bažant (2000, 2002) and the present discussion, Gao et al. (1999a) and Huang et al. (2000) also found that the use of couple stress τ_{ijk} offers no advantage and causes an unnecessary complication in the MSG theory. Realizing this, they soon modified the MSG theory by removing the couple stress and named the modification as the "TNT of plasticity". The strain gradient η_{ijk} is in the TNT theory treated by a nonlocal approximation (Gao and Huang, 2001).

4.1. Formulation of TNT theory

In TNT theory, the strain gradient is introduced as an integral-type nonlocal variable. To this end, Gao and Huang (2001) expand the strain component ϵ_{ij} into Taylor series in the neighborhood of point $\boldsymbol{x}$

$$\epsilon_{ij}(\boldsymbol{x}+\boldsymbol{\xi}) = \epsilon_{ij}(\boldsymbol{x}) + \epsilon_{ij,m}\xi_m + \mathrm{O}(|\xi|^2) \tag{94}$$

where $\boldsymbol{\xi}$ denotes a local coordinate centered at $\boldsymbol{x}$. Then they multiply this expression with ξ_k over the volume of a small representative cell V_{cell} which contains $\boldsymbol{x}$ and is sufficiently small,

$$\int_{V_{\text{cell}}} \epsilon_{ij}(\boldsymbol{x}+\boldsymbol{\xi})\xi_k \,\mathrm{d}V = \epsilon_{ij}(\boldsymbol{x}) \int_{V_{\text{cell}}} \xi_k \,\mathrm{d}V + \epsilon_{ij,m} \int_{V_{\text{cell}}} \xi_k \xi_m \,\mathrm{d}V \tag{95}$$

Therefore, the gradient term $\epsilon_{ij,k}$ can be approximated by an integral of strain ϵ

$$\epsilon_{ij,k} = \int_{V_{\text{cell}}} [\epsilon_{ij}(\boldsymbol{x}+\boldsymbol{\xi}) - \epsilon_{ij}(\boldsymbol{x})]\xi_m \,\mathrm{d}V \left(\int_{V_{\text{cell}}} \xi_k \xi_m \,\mathrm{d}V \right)^{-1} \tag{96}$$

If V_{cell} is a cube centered at $\boldsymbol{x}$,

$$\epsilon_{ij,k} = \frac{1}{I_\epsilon}\int_{V_{\text{cell}}} \epsilon_{ij}\xi_k \, \mathrm{d}V \quad \text{with } I_\epsilon = \int_{V_{\text{cell}}} \xi_1^2 \, \mathrm{d}V = \frac{1}{12} l_\epsilon^5 \tag{97}$$

Thus the strain gradient η_{ijk}, defined as $u_{k,ij}$, can be treated as a nonlocal variable:

$$\eta_{ijk} = \frac{1}{I_\epsilon}\int_{V_{\text{cell}}} [\epsilon_{ik}\xi_j + \epsilon_{jk}\xi_i - \epsilon_{ij}\xi_i] \, \mathrm{d}V \tag{98}$$

Using this formula, one can easily define $\boldsymbol{\eta}^{\text{H}}$, $\boldsymbol{\eta}'$ and the effective strain gradient $\eta = (1/2)(\eta'_{ijk}\eta'_{ijk})^{1/2}$, which is the same as that defined in the MSG theory. For cells of other shapes and for points near the boundary, the general form (96), rather than (97), should of course be used to define the nonlocal variable η.

The constitutive relation of TNT theory can be written as:

$$\sigma_{ik} = K\delta_{ik}\epsilon_{nn} + \frac{2\sigma}{3\epsilon}\epsilon'_{ik} \tag{99}$$

where

$$\sigma = \sigma_{\text{ref}}\sqrt{f^2(\epsilon) + l\eta} \tag{100}$$

Although this expression for stress intensity σ is similar to that used in the MSG theory (Eq. (51) for $p = 1$, $q = 2$), σ_{Y} is replaced by σ_{ref}, representing any reference stress in uniaxial tension. The choice of σ_{ref} will affect the value of the corresponding material length l (Gao and Huang, 2001, however, show the constitutive equation of the TNT theory to be independent of the choice of σ_{ref}). Similarly to the MSG theory, we can also consider a more general relation here: $\sigma = \sigma_{\text{ref}}[f^q(\epsilon) + (l\eta)^p]^{1/q}$ with arbitrary positive exponents p and q (Gao et al.'s TNT theory corresponds to the case $p = 1$, $q = 2$).

Obviously the constitutive equation is similar to that of the MSG theory, although the effective strain gradient is now defined as a nonlocal variable, and the couple stress has disappeared. If we set $\beta = 0$, we can find that the TNT theory is almost the same, except that, in the TNT theory, the strain gradient is defined as a nonlocal variable. So the size effect analysis of the TNT theory will be similar to that in Section 3.5.

The only major difference between the TNT theory and the classical plasticity theory is that the constitutive equation of the TNT theory is nonlocal and size-dependent (because an intrinsic material length l is involved). The differential equation of equilibrium of the TNT theory is the same as the classical one, i.e. $\sigma_{ik,i} + f_k = 0$.

4.2. Size effect analysis

It can be shown that the integral in (97) is in fact equivalent to a finite difference representation of a derivative of ϵ_{ij} (Gao and Huang, 2001). So we define dimensionless variables similar to (19) and (20):

$$\bar{x}_i = x_i/D, \quad \bar{u}_i = u_i/D, \quad \bar{\epsilon}_{ij} = \epsilon_{ij}, \quad \bar{\eta}_{ijk} = \eta_{ijk}D, \quad \bar{\epsilon} = \epsilon \tag{101}$$

$$\bar{\eta} = \eta D, \quad \bar{\sigma}_{ik} = \sigma_{ik}/\sigma_{\text{ref}}, \quad \bar{\sigma} = \sigma/\sigma_{\text{ref}}, \quad \bar{f}_k = f_k D/\sigma_{\text{N}} \tag{102}$$

The derivatives with respect to dimensionless coordinates $\bar{x}_i$ are again denoted as $\partial_i = \partial/\partial\bar{x}_i$. The constitutive law of the TNT theory may now be rewritten as:

$$\bar{\sigma}_{ik} = \frac{K}{\sigma_{\text{ref}}}\delta_{ik}\bar{\epsilon}_{nn} + \frac{2\bar{\sigma}}{3\bar{\epsilon}}\bar{\epsilon}'_{ik} \tag{103}$$

and the differential equations of equilibrium transform as

$$\partial_i\bar{\sigma}_{ik} + \frac{\sigma_{\text{N}}}{\sigma_{\text{ref}}}\bar{f}_k = 0 \tag{104}$$

Restricting attention, as before, to the loading in the form of body forces, we may again avoid the formulation of boundary conditions, considering them as homogeneous.

For $l/D \to 0$, the constitutive equation (99) of the TNT theory turns out to be the same as in the classical plasticity theory, in which there is no size effect.

The opposite asymptotic behavior for $D/l \to 0$ is a little more complex. From (51) we have $\bar{\sigma} \approx (\bar{\eta} l/D)^{p/q}$ when $D/l \to 0$. After substituting (103) into (55), we obtain the differential equations of equilibrium in the form

$$\partial_i \left[\frac{K}{\sigma_Y} \delta_{ik} \bar{\epsilon}_{nn} + \frac{2}{3\bar{\epsilon}} \left(\frac{l\bar{\eta}}{D} \right)^{p/q} \bar{\epsilon}'_{ik} \right] = -\frac{\sigma_N}{\sigma_Y} \bar{f}_k \tag{105}$$

When $D \to 0$, the second term on the left hand side will dominate, and so the asymptotic form of the field equation is:

$$\partial_i \left(\bar{\eta}^{p/q} \frac{\bar{\epsilon}'_{ijk}}{\bar{\epsilon}} \right) = \chi \bar{f}_k, \quad \text{with } \chi = -\frac{3}{2} \frac{\sigma_N}{\sigma_Y} \left(\frac{D}{l} \right)^{p/q} \tag{106}$$

where χ is a size independent constant. Thus the small-size asymptotic scaling law for the TNT theory turns to be, in general, $\sigma_N = -\sigma_Y \chi (l/D)^{p/q}$, and according to Gao et al.'s theory ($p/q = 1/2$):

$$\sigma_N \propto D^{-1/2} \tag{107}$$

4.3. Small-size asymptotic load–deflection response

The small-size asymptotic load–deflection response of the TNT theory is also the same as the case of $\beta = 0$ in MSG theory. When D and w are both sufficiently small, the dominant term on the left-hand side of (105) will be the second one; hence, in general, $\bar{f}_k \propto w^{p/q}$ and, for Gao et al.'s theory:

$$\bar{f}_k \propto w^{1/2} \tag{108}$$

Again, the elastic response is missing in the asymptotic case.

5. Asymptotic-matching approximation

The small-size asymptotic scaling law established in Bažant (2000, 2002) for the MSG theory, and here for the MSG and Fleck and Hutchinson's theories, can be closely approached when the cross-section size of a metallic specimen is reduced to about 10 nm. This size is of course too small for these theories to be valid. A realistic theory for such a small size would have to take into account the surface energy and surface tension effects, and it would have to be based on an interatomic potential. Does it mean that the asymptotic scaling laws that have been established are only of academic interest? Certainly not, for three reasons.

One reason simply is that a theory that implies an unreasonable asymptotic size effect should better be avoided. For the scaling of nominal stress, power laws with exponents −5/2 or −2 would in principle be impossibly strong, far stronger than any known size effects in solid mechanics. Even a power law with the exponent $-(n+1)/n$, which could typically be about −1.3, seems to be somewhat too strong.

The second reason is that knowledge of two opposite asymptotic behaviors, normally those for the large and small scales, makes it possible to benefit from asymptotic matching. Asymptotic matching is a broad term for mathematical techniques that yield approximate solutions for the practical length or size range of interest in which the solution is much harder than it is the adjacent asymptotic ranges. The earliest example of asymptotic matching is the boundary layer theory in fluid mechanics, conceived by Prandtl (1904). Ever since, the asymptotic matching has been pursued systematically in fluid mechanics (Bender and Orszag, 1978; Barenblatt, 1979; Hinch, 1991), but in the solid mechanics community this technique has not been exploited until the recent researches of size effect (Bažant, 1997, 1999).

The third reason is the gradual transition of response to the asymptotic one. For example, we found the small-size asymptotic load–deflection diagram for the MSG theory to have a positive curvature, which is unreasonable. Therefore, as the size is reduced from, say, $D = 100$–0.1 μm, one must expect the curvature of this diagram to change gradually from a large negative value to a small negative value, and then to a small positive value. The comparisons of the MSG theory with experiments, exhibited in Gao et al. (1999a,b), demonstrate this kind of transition and one may note that, for the smallest sizes tested (about 1 μm), the theory predictions for micro-bending are too soft at small stress and too stiff at large stress, compared to test data.

In the special case of scaling laws for geometrically similar bodies, the asymptotic matching is much simpler than it is in fluid mechanics. It does not necessitate any solution of differential equations, in contrast to the boundary layer theory. Rather, it suffices to find a simple and smooth formula that has the required small-size and large-size asymptotic properties. This simple kind of asymptotic matching has been systematically pursued with success during the last two decades for the purpose of establishing size effect laws for various types of failure of quasibrittle materials—first for concrete, and rocks, and more recently for sea ice and fiber composites (Bažant and Planas, 1998; Bažant and Chen, 1997; Bažant and Novák, 2000; Bažant, 1984, 1997, 1999). A similar approach was suggested in Bažant (2000, 2002) to be taken for the gradient plasticity of metals on the micrometer scale. Solving this problem for the practical range of interest (0.1–100 μm) would be hard but the asymptotic cases are easy to determine for all the available theories, as shown in Bažant (2000, 2002) and in this paper.

In the present case, the asymptotic matching approach calls for a formula that yields a smooth transition between the case of no size effect for $D \to \infty$ and the case of power law $\sigma_N \propto D^{-s}$ for $D \to 0$ ($s > 0$). Perhaps the simplest formula with these properties is

$$\sigma_N = \sigma_0 \left[1 + \left(\frac{D_0}{D} \right)^{2s/r} \right]^{r/2} \tag{109}$$

where r is a constant, which determines how slow the transition is. The larger is r, the slower is the transition. For the case of the MSG theory, $s = 5/2$ (Bažant, 2002) or 2, or 3/2; for Fleck et al.'s theories, $s = (n+1)/n$; and for the TNT theory, $s = 1/2$. The parameters σ_0 and D_0 can be determined as follows

$$\sigma_0 = \lim_{D/l \to \infty} \sigma_N, \quad D_0 = \left[\lim_{D/l \to 0} \left(\frac{\sigma_N D^s}{\sigma_0} \right) \right]^{1/s} \tag{110}$$

Parameter r, which controls the slowness of the transition, and it must be calibrated by experiments or numerical simulations.

6. Comparison of asymptotic scaling laws

6.1. Asymptotic behavior

For Fleck and Hutchinson's CS and SG theories, we have

$$\sigma_N \propto D^{-r}, \quad r = \frac{n+1}{n} \quad (1 < r \leqslant 2) \tag{111}$$

For the MSG theory, we have

$$\sigma_N \propto D^{-s} \quad s = \tfrac{5}{2} \text{ in general, } s = 2 \text{ or } \tfrac{3}{2} \text{ in special cases} \tag{112}$$

and if, as suggested in Bažant (2000, 2002) and again here, $\beta = 0$, then $s = 1/2$.

For the TNT theory, we have (with $p/q = 1/2$):

$$\sigma_N \propto D^{-1/2} \tag{113}$$

6.2. Small-size asymptotic load–deflection response

For Fleck and Hutchinson's CS and SG theories, we have

$$\bar{f}_k \propto w^{1/n}, \quad 1/n \leqslant 1 \tag{114}$$

where n is the strain hardening exponent.

For the MSG theory, in general, we have

$$\bar{f}_k \propto w^{3/2} \quad \text{and} \quad \bar{f}_k \propto w^{1/2}, \quad \text{for } w \ll D \tag{115}$$

In some special cases, though, the scaling can be $\bar{f}_k \propto w^{1/2}$ regardless of the value of w/D.

For the TNT theory, we have $\bar{f}_k \propto w^{p/q}$, and for Gao et al.'s formulation ($p/q = 1/2$):

$$\bar{f}_k \propto w^{1/2} \tag{116}$$

Note again that the elastic part of response asymptotically vanishes for (115) and (116).

6.3. Linkage between the small-size asymptotic behavior and the constitutive relation for the macro-scale

In the MSG and TNT theories, the small size asymptotic behaviors are determined by micro-scale material parameters which are independent of the macro-scale material parameters. However, in Fleck and Hutchinson's theories, the small-size asymptotic behavior is determined solely by macro-scale material parameters, which might be a somewhat questionable aspect of these theories.

6.4. Existence of strain energy density function

An advantage of Fleck and Hutchinson's CS and SG theories is that the strain energy density function exists. It is defined as a function of the so-called "combined strain quantity" $\mathscr{E}$, which in turn is a variable depending on $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$.

For the MSG theory, the strain energy density function does not exist because the reciprocity relation is not met; see (87).

For the TNT theory, the strain energy density function can be defined according to its constitutive equation (99) in a way similar to the classical theory of plasticity. The difference is that the hardening is defined as a function of the nonlocal strain.

7. Scaling of plastic hardening modulus in Acharya and Bassani's gradient theory

Finally, a brief look at the theory of Acharya and Bassani (2000) and Bassani (2001) is appropriate. These authors developed a simple gradient theory which differs significantly from the previous four theories. It is a generalization of the classical incremental theory of macro-scale plasticity, rather than the deformation (total strain) theory. In contrast to the previous four theories, in which the strain-gradient tensor $\boldsymbol{\eta}$ is defined as a third-order tensor representing the gradient of total strain, the lattice incompatibility is measured by a second-order tensor defined by the following contraction of the gradient of plastic strain ϵ_{ij}^p:

$$\alpha_{ij} = e_{jkl}\epsilon_{il,k}^p \tag{117}$$

where e_{jkl} is the alternating symbol. The plastic hardening is assumed to be governed by the invariant:

$$\alpha = \sqrt{2\alpha_{ij}\alpha_{ji}} \tag{118}$$

Then the basic equations of the classical J_2 flow theory are modified as follows:

$$\tau = \sqrt{\frac{\sigma'_{ij}\sigma'_{ij}}{2}} = \tau_{\rm cr}, \quad \dot{\tau} = \dot{\tau}_{\rm cr} = h(\gamma^p, \alpha)\dot{\gamma}^p \tag{119}$$

$$\dot{\epsilon}_{ij}^p = \left(\frac{\dot{\gamma}^p}{2\tau}\right)\sigma'_{ij}, \quad \dot{\sigma}_{ij} = C_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p), \quad \gamma^p = \sqrt{\frac{2}{3}\epsilon_{ij}^p\epsilon_{ij}^p} \tag{120}$$

The variables used in the above are almost the same as in the classical J_2 flow theory except that the instantaneous hardening-rate function h depends not only on plastic strain invariant γ^p but also on α. The following hardening function $h(\gamma^p, \alpha)$ is used by Bassani (2001) for numerical simulation of the microtorsion test:

$$h(\gamma^p, \alpha) = h_0\left(\frac{\gamma^p}{\gamma_0} + 1\right)^{N-1}\left[1 + \frac{l^2(\alpha/\gamma_0)^2}{1 + c(\gamma^p/\gamma_0)^2}\right]^{1/2} \tag{121}$$

where l is a material length introduced for dimensionality reasons, and h_0, γ_0, c and N are further material constants (all positive).

Although a full analysis of scaling of this theory is beyond the scope of this paper, some simple observations can be made. From the scale transformations $\bar{u}_i = u_i/D$, $\bar{\epsilon}_{il,k} = \epsilon_{il,k}D$, it follows that $\alpha_{ij} = \bar{\alpha}_{ij}/D$, where the overbars again denote the dimensionless variables. Since $\bar{\gamma}^p = \gamma^p$, the plastic hardening modulus defined by (121) scales for $D \to 0$ as

$$h(\gamma^p, \alpha) = h_0\left(\frac{\gamma^p}{\gamma_0} + 1\right)^{N-1}[1 + c(\gamma^p/\gamma_0)^2]^{-1/2}\frac{\bar{\alpha}}{\gamma_0}\frac{l}{D} \propto D^{-1} \tag{122}$$

This means that, at the same strain level, the slope of the plastic hardening curve increases as D^{-1} when $D \to 0$. When the plastic strain becomes much larger than the elastic strain, and when the strain distributions and history are similar, then of course the nominal stress $\sigma_{\rm N}$ must also scale asymptotically as D^{-1}. This is again a curiously strong asymptotic size effect, not much less strong than that found for the MSG and CS theories. Even though this excessive size effect is approached only outside the range of applicability of the theory, one must expect that it would impair the representation of test data in the middle range of practical interest.

The excessive asymptotic size effect could be avoided by redefining the plastic hardening modulus in (121) as follows:

$$h(\gamma^p, \alpha) = h_0\left(\frac{\gamma^p}{\gamma_0} + 1\right)^{N-1}\left[1 + \frac{l\alpha/\gamma_0}{1 + c(\gamma^p/\gamma_0)}\right]^{1/2} \tag{123}$$

With this revision, which should be checked against test data, the asymptotic scaling would become

$$h(\gamma^p, \alpha) \propto D^{-1/2} \quad \text{when } D \to 0 \tag{124}$$

which seems more reasonable and similar to Bažant's (2000) proposal for revision of MSG theory, as well as to the TNT theory.

8. Summary and conclusions

(1) The approach introduced by Bažant (2000, 2002) for determining the asymptotic properties of Gao et al.'s (1999a,b) MSG theory of metal plasticity is now applied to Gao and Huang's (2001) newer theory, the TNT, and to Fleck and Hutchinson's (1993, 1997) theories, the original gradient theories of metal plasticity on the micrometer scale. The small-size asymptotic scaling laws and load–deflection diagrams of these two theories are determined. Furthermore, Bažant's (2000, 2002) asymptotic analysis of the MSG theory is extended to two special cases with atypical asymptotic scaling.

(2) The small-size asymptotic scaling laws for the nominal stress σ_N in all the existing theories are power laws, but there are wide disparities among them. For the MSG theory, Bažant (2000, 2002) showed that, in general, $\sigma_N \propto D^{-5/2}$, which is an unreasonably strong size effect. For two special cases of the MSG theory it is shown here that $\sigma_N \propto D^{-2}$ and $D^{-3/2}$, which is also very strong. For the classical Fleck and Hutchinson CS and SG theories, it is found that $\sigma_N \propto D^{-(n+1)/n}$ (where n is the exponent of the strain hardening law on the macro-scale); typically, $\sigma_N \propto D^{-1.3}$, which is also quite strong. For the TNT theory (as well as for the modification of the MSG theory proposed in Bažant, 2000, 2002), $\sigma_N \propto D^{-1/2}$, which seems reasonable.

(3) The small size asymptotic load–deflection diagram of the MSG theory was shown in Bažant (2000, 2002) to be a power law of the type $\sigma_N \propto w^{3/2}$ (w is the deflection). The fact that the slope of this diagram is initially horizontal and then increases, rather then decreases, is unrealistic. For the Fleck and Hutchinson theories, it is shown here that $\sigma_N \propto w^{1/n}$, which is typically about $w^{1/2}$ and is reasonable overall, except that the initial slope is vertical (i.e., the initial stiffness is infinite, elasticity vanishes). The same result, $\sigma_N \propto w^{1/2}$, is obtained here for the TNT theory.

(4) Although the small-size asymptotic behavior is closely approached only at sizes much smaller than the range of applicability of the strain-gradient theories of plasticity (which is about 0.1–100 μm), the knowledge of this behavior is useful for developing asymptotic matching approximations for the realistic middle range.

(5) A simple asymptotic formula for the asymptotic matching of the small-size and large-size behaviors, proposed in Bažant (2000, 2002) for the MSG theory, is here extended to the other theories and is shown to provide good approximations of the experimental as well numerical results for the middle range. The availability of such formulae means that the stress analysis for the middle range, which is much more difficult than the asymptotic analysis, can be avoided. Such an approach, however, is possible only if the small-size asymptotic behavior is realistic. This is for example documented by the fact that, for the MSG theory, the response for sizes under 1 μm is somewhat too soft at small deflections and too stiff at large deflections, and that the size effect at the lower limit of the range of experiments is excessive. The detrimental consequence of unrealistic small-size asymptotic properties is that the possibility of asymptotic matching approximations is lost.

(6) The plastic hardening modulus in the theory of Bassani and Acharya scales asymptotically as D^{-1}, which also seems excessive. However, a simple modification can achieve the scaling to be $D^{-1/2}$.

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